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On the Expansion of Functions in Infinite Series.

By W. H. Echols.

Cauchy's methods leave little to be desired in the expansion of functions in infinite series, and have transferred all at once the treatment of the subject from the methods of the Differential to those of the Integral Calculus. This is justly so, on account of the increased rigor of the analysis applied and on account of the facilities which it offers for the study of the properties of functions. Yet it has seemed to me not uninteresting to seek generalization of the older methods and to question wherein they have failed to supply the demand made of them. The following is a brief outline sketch made along the lines of enquiry, and imperfect as it may appear here and there in points but lightly touched, it seems to throw a light upon this subject from a different source, and it would appear that there may be certain points of vantage in the serial coefficients over their integral forms in discussing certain properties of functions.

1. Let fz and $\phi_r z\, (r = 1 \ldots . n - 1)$ be monogenic functions of a complex variable z, which are holomorphous throughout a certain region containing the arbitrary closed path C.

Let there be a function

$$S_n z \equiv \sum_{r=0}^{n-1} A_r \phi_r z,$$

in which $\phi_0 z \equiv 1$ and $\phi_r z$ are functions whose law of successive formation with respect to r is known, and whose coefficients A_r, independent of the variable z, are at present arbitrary, but whose law of formation is to be determined.

Let $R_1, \ldots, R_n$ be the differences between the functions fz and $S_n z$ at the points $z_1, \ldots, z_n$. We then have in

$$fz_m = S_n z_m + R_m, \qquad (m = 1 \ldots . n)$$

n relations connecting the n values A_r $(r = 0 \ldots\ldots n - 1)$. Let us so determine these n quantities A_r that we shall have $R_m = 0$. This condition, which assures the arithmetical coincidence of the functions fz and $S_n z$ at the points $z_{1 \ldots\ldots n}$, is expressed by*

$$A_r = \frac{|\phi_0 z_1 \ldots\ldots \phi_{r-1} z_r . f z_{r+1} . \phi_{r+1} z_{r+2} \ldots\ldots \phi_{n-1} z_n|}{|\phi_0 z_1 \ldots\ldots \phi_{n-1} z_n|} . \tag{1}$$

Let the n points $z_{1 \ldots\ldots n}$ be on the path C so that these points are at the vertices of an inscribed polygon of n sides $w_1 \ldots\ldots w_n$. We then have generally

$$z_{r+1} - z_r = w_r \quad \text{and} \quad z_1 + \sum_1^n w_r = z_1 .$$

We suppose w_r $(r = 1 \ldots\ldots n)$ to be such that mod w_r vanishes along with $1/n$. The condition that fz and $S_n z$ shall coincide at an infinite number of consecutive points along C, that is, all along the path C, is that the coefficient A_r shall have the limiting value of this ratio (1) when $n = \infty$. This ratio takes the indeterminate form $0/0$ when $n = \infty$, since then w_r is infinitesimal and z_p and z_{n-p} converge to z_1. To remove this indetermination we apply to each term of the ratio the operator

$$\left(\frac{d}{dz_2}\right)^1_{z_2 = z_1} \ldots\ldots \left(\frac{d}{dz_n}\right)^{n-1}_{z_n = z_1} . \tag{2}$$

Whence

$$A_r = \mathop{\mathcal{L}}_{n=\infty} \frac{|\phi_1' z_1 \ldots\ldots \phi_{r-1}^{r-1} z_1 . f^r z_1 . \phi_{r+1}^{r+1} z_1 \ldots\ldots \phi_{n-1}^{n-1} z_1|}{|\phi_1' z_1 \ldots\ldots \phi_{n-1}^{n-1} z_1|} , \tag{3}$$

and its limit may be taken without indetermination.

Let z be an arbitrary point taken anywhere on C at a finite distance from z_1. The condition (3) is independent of the route pursued in tracing the closed path C through z_1 and z, since the intermediate points no longer appear and the derivatives of the monogenic functions at z_1 are independent of the direction with which C passes through z_1, and since the functions are supposed uniform, their values at z_1 and z are also independent of the path traced between z_1 and z. This being so, condition (3) expresses, therefore, also the more general condition that the functions fz and $S_\infty z$ shall coincide at all points throughout the whole region of holomorphism. If, therefore, the condition (3) be found to be such that $S_\infty z$ is a convergent series, then the arithmetical equivalence of this series with the function fz at all points throughout the holomorphic region is assured.

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*The vertical space rule being used here to denote the umbral determinant notation and not the modulus of the complex quantity. The terms between the verticals being those of the principal diagonal.

For example, if $\phi_r z \equiv z^r/r!$, the value of $S_\infty z$ is readily seen to be

$$\sum_0^\infty (z - z_1)^r f^r z_1/r!,$$

the convergence of which is assured when the derivatives of finite order of fz at z_1 are not infinite, and after some fixed m^{th} derivative the $(m+p)^{\text{th}}$ derivative is not greater than the $(m+p)^{\text{th}}$ power of a finite quantity. If we may identify such properties to be those of a holomorphic function, then every holomorphic function is expansible in Taylor's series throughout its region of holomorphism. Let us regard the functions with which we are going to deal as enjoying these properties, then all properties which may be deduced from Taylor's series belong to them.

2. If we stop this process at the n^{th} term we obtain an interpolation formula for fz along the chosen path C expressed in terms of the functions at the points $z_{1 \dots n}$ along C.

The difference R between the function fz and the interpolation value $S_n z$ at any arbitrary running point z on C will be

$$\frac{|fz,\ \phi_0 z_1 \dots \phi_{n-1} z_n|}{|\phi_0 z_1 \dots \phi_{n-1} z_n|} = R. \tag{4}$$

We may, if we choose, regard the points $z_{1 \dots n}$ as being arbitrarily located isolated points about in the holomorphic region, and z as before a wandering point in that region. If we wish to evaluate this ratio when $n = \infty$, we may remove the indetermination caused by the convergence of the z's to z_1 by applying the operator (2) to each term. Regarding the z's as shrunk upon z_1, we have without indetermination the interpolation

$$\frac{\begin{vmatrix} fz & 1 & \phi_1 z & \dots\, \phi_{n-1} z \\ fz_1 & 1 & \phi_1 z_1 & \dots\, \phi_{n-1} z_1 \\ f' z_1 & 0 & \phi_1' z_1 & \dots\, \phi_{n-1}' z_1 \\ \dots & \dots & \dots & \dots \\ f^{n-1} z_1 & 0 & \phi_1^{n-1} z_1 & \dots\, \phi_{n-1}^{n-1} z_1 \end{vmatrix}}{|\phi_1' z_1 \dots \phi_{n-1}^{n-1} z_1|} = R, \tag{5}$$

or

$$fz = \sum_{0}^{n-1} A_r \phi_r z + R.$$

*3. Let ψz be a function having the same properties with fz, and let us seek to compare their interpolative differences as obtained from (5), which we call R_ψ and R_f.

If Fz be a holomorphous function throughout a connected region containing the two points z' and z'', and if these points be zeros of Fz, then the derivative of Fz will have a zero somewhere on a finite path traced in this region between z' and z'', by the theorem of mean value

$$Fz' - Fz'' = \lambda S F'\zeta,$$

mod $\lambda \leqq 1$, $S =$ length of path, and ζ a point on this path between z' and z''.

Let Z be the ratio of R_f to R_ψ, which is to be determined. Let z_0 be some arbitrarily chosen fixed point on C. Then we have $Z_0 = R_{f_0}/R_{\psi_0}$. Consider the function

$$R_f - Z_0 R_\psi .$$

This and its first $n - 1$ derivatives have the zero z_1. The function also has the zero z_0, therefore if the theorem of mean value holds good for it and its first $n - 1$ derivatives, its n^{th} derivative must have a zero ζ somewhere on C. Whence

$$D^n_{z=\zeta} R_f - Z_0 D^n_{z=\zeta} R_\psi = 0,$$

and if the n^{th} derivative of R_ψ does not vanish in the region considered we may divide by it and have

$$R_f / R_\psi = D^n_{z=\zeta} R_f / D^n_{z=\zeta} R_\psi,$$

since we may drop the subscript, z_0 being an arbitrary point. If, therefore, we be examining the expansion of an unfamiliar function fz, we may investigate it through the aid of a more familiar one, ψz. The remainder after the n^{th} term of the interpolation being

$$R_f = R_\psi \frac{D^n_{z=\zeta} R_f}{D^n_{z=\zeta} R_\psi}. \qquad (6)$$

If ψz be expansible to infinity, then R_ψ vanishes with $1/n$ and we are concerned with the limit, when $n = \infty$, of the ratio of the n^{th} derivatives of

$$fz - \sum_{0}^{n-1} A_r \phi_r z \quad \text{and} \quad \psi z - \sum_{0}^{n-1} A_r \phi_r z$$

in the region under consideration.

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§3 applies only to functions of a real variable. $\lambda \equiv 1$.

If we take $\psi z \equiv \phi_n z$, then in the example given above of Taylor's series we have

$$R_f = \frac{(z - z_1)^n}{n!} f^n \zeta,$$

$$= \lambda^n S^n f^n \zeta / n!$$

Any number of apparently more general forms of this remainder can be made from the general theorem (6). Thus put R_f^n and R_ψ^p for the remainders after n and p terms of expansion of fz and ψz. Then

$$R_f = R_\psi^p \frac{D_{z=\zeta'}^p R_{\phi_p}^p}{D_{z=\zeta}^p R_{\phi_n}^n} \frac{D_{z=\zeta}^n R_f^n}{D_{z=\zeta'}^n R_\psi^p} \frac{R_{\phi_n}^n}{R_{\phi_p}^p}.$$

Let $\phi_r z = (z - z_1)^r / r!$ and $\psi z = (z - z_1)^q$, $q > p$. Then

$$R_f = \frac{p!}{n!} \frac{(z - z_1)^{n+q-p} f^n \zeta}{q \ldots . (q - p + 1)(\zeta' - z_1)^{q-p}}.$$

4. It is one of the properties of a holomorphic function deducible from Taylor's formula, that if such a function vanishes all along a line of finite length it vanishes throughout its holomorphic region. The limit (3) is this condition which, if fulfilled by the convergence of $S_\infty z$, secures the coincidence of fz with this series throughout this region.

The determination of the coefficients of $S_\infty z$ involves the determination of the limit of the ratio of the first minor of a Wronskian to its major when the number of rows is infinite. This is easily evaluated in two general cases; first, when the elements of the Wronskian on either side of the principal diagonal vanish; second, when the Wronskian becomes a difference-product.* It is desirable to know what other forms admit of evaluation.

5. A class of ϕ_r functions of importance, such as certain periodic functions, are holomorphic for finite values of r, but cease to be so for infinite values of r. In general, if it be merely a question of arithmetical equivalence between fz and $S_\infty z$, within a certain boundary, it will be sufficient to show that R is infinitesimal for the region considered. If we enquire further into the nature of the equivalence and the extent of the coincidence, it will be necessary to investigate the relation between the infinitesimals R and w, which, if of the orders i and j respectively, we say that the function fz and the series $S_\infty z$ have a coincidence of the order $(i - j + 1)$ with contact of the order $(i - j)$.

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* Annals of Mathematics, Vol. 7, No. 4, 5.